Inequality 15

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Q) Let $a, b, c \in \mathbb{R}^+$, then prove that,

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

Ans:- $$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

$$\Leftrightarrow 1+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1 \geq 2+\frac{2(a+b+c)}{\sqrt[3]{abc}}$$

$$\Leftrightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{c}{b}+\frac{b}{a} \geq \frac{2(a+b+c)}{\sqrt[3]{abc}}$$

$a = x^3$, $b = y^3$, $c = z^3$ (let)

$$\Leftrightarrow \frac{x^3}{y^3}+\frac{y^3}{z^3}+\frac{z^3}{x^3}+\frac{x^3}{z^3}+\frac{z^3}{y^3}+\frac{y^3}{x^3} \geq \frac{2(x^3+y^3+z^3)}{xyz}$$

Let $\frac{x}{y}, \frac{y}{z}, \frac{z}{x}, \frac{x}{z}, \frac{z}{y}, \frac{y}{x}$ be in some order — ①

Then $\frac{x^2}{y^2}, \frac{y^2}{z^2}, \frac{z^2}{x^2}, \frac{x^2}{z^2}, \frac{z^2}{y^2}, \frac{y^2}{x^2}$ will be in same order — ②

But, $\frac{y}{z}, \frac{z}{x}, \frac{x}{y}, \frac{z}{y}, \frac{y}{x}, \frac{x}{z}$ will be in different order — ③

so we get ② with ① $\geq$ ② with ③ in rearrangement inequality

$$\Rightarrow \frac{x^3}{y^3}+\frac{y^3}{z^3}+\frac{z^3}{x^3}+\frac{x^3}{z^3}+\frac{z^3}{y^3}+\frac{y^3}{x^3} \geq 2\left(\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}\right)$$

$$= 2\left(\frac{x^3+y^3+z^3}{xyz}\right)$$

Q) $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$ and $\sum_{i=1}^{n} a_i = 1$, then prove that,

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{1-a_i}} \geq \frac{1}{\sqrt{n-1}} \sum_{i=1}^{n} \sqrt{a_i}$$

$AM \geq GM$

$\frac{1}{AM} \leq \frac{1}{GM}$

$AM^n \geq GM^n$

$\frac{1}{1} \times \frac{1}{2} = \frac{1}{1 \times 2}$

Ans:- $\frac{a_i}{\sqrt{1-a_i}} = \frac{1}{\sqrt{1-a_i}} - \sqrt{1-a_i}$

$$AM \geq GM \xrightarrow[\text{here}]{\text{from}} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{1-a_i}} \geq \sqrt[n]{\prod_{i=1}^{n} \left(\frac{1}{\sqrt{1-a_i}}\right)}$$

idea $\left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{n}} = \left(\frac{1}{2^{1/n}}\right)^{\frac{1}{2}}$

$$= \sqrt[n]{\frac{1}{\prod_{i=1}^{n} \sqrt{1-a_i}}}$$

idea

$$= \sqrt{\frac{1}{\sqrt[n]{\prod_{i=1}^{n} (1-a_i)}}}$$

$\frac{1}{AM} \leq \frac{1}{GM}$

$\sqrt{\frac{1}{AM}} \leq \sqrt{\frac{1}{GM}} \xrightarrow[\text{here}]{\text{from}}$

$$\geq \sqrt{\frac{1}{\frac{1}{n}\sum_{i=1}^{n} (1-a_i)}}$$

$$= \sqrt{\frac{n}{\sum_{i=1}^{n} (1-a_i)}}$$

$\sum_{i=1}^{n} (1-a_i)$

$= \sum_{i=1}^{n} (1) - \sum_{i=1}^{n} (a_i)$

$= n - 1$

$$= \sqrt{\frac{n}{n-1}}$$

Q) $x_1, x_2, \ldots, x_n > 0$ such that $\frac{1}{1+x_1} + \cdots + \frac{1}{1+x_n} = 1$

$$x_1 x_2 \cdots x_n \geq (n-1)^n$$

$y_i = \frac{1}{\ldots}$ $\quad x_i + 1 = \frac{1}{y_i}$ $\quad x_i = \frac{1}{y_i} - 1$

Ans:— $y_i = \frac{1}{1+x_i}$ $x_i + 1 = \frac{1}{y_i}$. $x_i = \frac{1}{y_i} - 1$

$$y_1 + y_2 + \cdots + y_n = 1 \Rightarrow \sum_{i=1}^{n} y_i = 1$$

AM $\geq$ GM

$$1 - y_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} (y_i) \geq (n-1) \left(\prod_{\substack{i=1 \\ i \neq j}}^{n} y_i \right)^{\frac{1}{n-1}}$$

$$x_1 x_2 \cdots x_n = \prod_{i=1}^{n} \left(\frac{1-y_i}{y_i} \right) = \frac{\prod_{i=1}^{n} (1-y_i)}{\prod_{i=1}^{n} (y_i)}$$

$$\geq \frac{(n-1)^n \prod_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} y_j \right)^{\frac{1}{n-1}}}{\prod_{i=1}^{n} (y_i)}$$

$$= \frac{(n-1)^n \left((y_1)^{\frac{1}{n-1}} \right)^{n-1} \left((y_2)^{\frac{1}{n-1}} \right)^{n-1} \cdots \left((y_n)^{\frac{1}{n-1}} \right)^{n-1}}{y_1 y_2 \cdots y_n}$$

$$= (n-1)^n$$

$i=1$, $(y_2 y_3 \cdots y_n)^{\frac{1}{n-1}}$

$i=2$, $(y_1 y_3 \cdots y_n)^{\frac{1}{n-1}}$

$\vdots$

$i=n$, $(y_1 y_2 \cdots y_{n-1})^{\frac{1}{n-1}}$

y_1 comes in $n-1$ terms
similarly for all other y_i's